# Lecture 17: Primal-Dual Interior-Point Methods for Conic Optimization

TTIC 31070 / CMSC 35470 / BUSF 36903 / STAT 31015

Convex Optimization

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Spring 2026

# Robust Linear Constraints → SOCP

**Robust constraint.** The nominal inequality $a_i^\top x \le b_i$ must hold even under ellipsoidal perturbation $a_i \mapsto a_i + P_i u$, $\|u\|_2 \le 1$:

$$\sup_{\|u\|_2 \le 1} (a_i + P_i u)^\top x \le b_i.$$

**Applications.** Production planning, portfolio optimization, and engineering design with uncertain coefficients.

**Compute the sup.** Since $\sup_{\|u\|_2 \le 1} u^\top P_i^\top x = \|P_i^\top x\|_2$,

$$a_i^\top x + \|P_i^\top x\|_2 \le b_i.$$

**Conic form.** Equivalently,

$$\boxed{\left(b_i - a_i^\top x,\ P_i^\top x\right) \in \mathcal{Q}^{k_i+1}, \quad \mathcal{Q}^{k+1} := \{(\tau, z) \in \mathbb{R} \times \mathbb{R}^k : \tau \ge \|z\|_2\}.}$$

# Max-Cut → SDP via Lifting

**Max-Cut.** $G = (V, E)$, weights $w_{ij} \geq 0$. Cut by signs $\varepsilon_i \in \{\pm 1\}$:

$$\max_{\varepsilon \in \{\pm 1\}^n} \sum_{\{i,j\} \in E} w_{ij} \, \frac{1 - \varepsilon_i \varepsilon_j}{2}. \quad \text{(nonconvex quadratic)}$$

**Why this matters.** Max-Cut is NP-hard, but appears in VLSI partitioning, statistical physics (Ising spin glasses), and community detection. We want a tractable surrogate.

**Lifting.** Set $X = \varepsilon\varepsilon^\top$. Then $X \succeq 0$, $X_{ii} = 1$, $\varepsilon_i \varepsilon_j = X_{ij}$.

**Drop rank-one.** SDP relaxation:

$$\boxed{\max \sum_{\{i,j\} \in E} w_{ij} \frac{1 - X_{ij}}{2} \quad \text{s.t.} \quad X_{ii} = 1 \; \forall i, \; X \succeq 0.}$$

# Goemans–Williamson Randomized Rounding

**From SDP solution back to a cut.** Let $X^\star \succeq 0$ with $X^\star_{ii} = 1$ solve the relaxation. Factor

$$X^\star = V^\top V, \qquad V = [v_1, \dots, v_n], \quad \|v_i\|_2 = 1, \ \langle v_i, v_j \rangle = X^\star_{ij}.$$

**Hyperplane rounding.** Sample $g \sim \mathcal{N}(0, I_n)$ and set $\hat{\varepsilon}_i := \operatorname{sign}(\langle g, v_i \rangle)$.

**Edge cut probability.** If $\theta_{ij} := \arccos(X^\star_{ij}) \in [0, \pi]$, rotational symmetry of $g$ in the 2D plane $\mathrm{span}(v_i, v_j)$ gives $\Pr[\hat{\varepsilon}_i \neq \hat{\varepsilon}_j] = \theta_{ij}/\pi$.

Theorem 17.1 (Goemans–Williamson, 1995) The expected cut weight satisfies

$$\mathbb{E}\Big[\sum_{\{i,j\} \in E} w_{ij} \tfrac{1 - \hat{\varepsilon}_i \hat{\varepsilon}_j}{2}\Big] \ \geq \ \alpha_{\mathrm{GW}} \cdot \mathrm{SDP}^\star \ \geq \ \alpha_{\mathrm{GW}} \cdot \mathrm{MaxCut}^\star,$$

where $\alpha_{\mathrm{GW}} = \min_{0 \leq \theta \leq \pi} \dfrac{2\theta}{\pi(1 - \cos\theta)} \ \approx \ 0.87856.$

# From Applications to Standard Conic Form

**Both examples share one pattern.** A nonlinear convex constraint is represented as **membership in a cone after a linear map**.

- Robust LP: $(b_i - a_i^\top x,\ P_i^\top x) \in \mathcal{Q}^{k_i+1}$.
- Max-Cut: $X \in \mathbb{S}_+^n$ with $X_{ii} = 1$.

**Point of conic form.** Not only uniform notation — it changes the objects the algorithm sees.

- Second-order cones keep robust Euclidean constraints as explicit vector inequalities.
- Semidefinite cones turn nonlinear quadratic relations into linear objectives over matrix variables.

Once the feasible set is written this way, *the right notion of a linear certificate is not a scalar multiplier but an element of the dual cone.*

# LP, SOCP, SDP Are Conic Programs

**Definition 17.2** (Standard conic programs). A conic linear program has the form

$$\inf\{\langle c, x\rangle : Ax = b,\ x \in K\},$$

where $K$ is a closed convex cone.

**LP** $K = \mathbb{R}^m_+$

**SOCP** $K = \mathbb{R}^p_+ \times \prod_i \mathcal{Q}^{k_i+1}, \quad \mathcal{Q}^{k+1} := \{(\tau, z) : \tau \geq \|z\|_2\}$

**SDP** $K = \mathbb{R}^p_+ \times \prod_j \mathbb{S}^{n_j}_+, \quad \mathbb{S}^n_+ := \{X \in \mathbb{S}^n : X \succeq 0\}$

**Takeaway.** SOCP and SDP are not different duality theories. They are conic programs with different choices of $K$.

# Dual Cone and Conic Primal-Dual Pair

**Definition 17.3** (Dual cone). For a cone $K \subseteq E$:

$$K^* := \{s \in E^* : \langle s, x \rangle \geq 0 \ \forall x \in K\}.$$

A cone is *proper* if it is closed, convex, pointed, with nonempty interior.

**Definition 17.4** (Conic primal-dual pair). Linear $A : E \to Y$, $b \in Y$, $c \in E^*$, closed convex cone $K \subseteq E$.

$$(P) \quad p^\star = \inf\{\langle c, x \rangle : Ax = b, \ x \in K\}.$$

$$(D) \quad d^\star = \sup\{\langle b, y \rangle : A^* y + s = c, \ s \in K^*\}.$$

$y \in Y^*$ is the equality multiplier; $s \in E^*$ is the dual-cone slack.

# Conic Weak Duality

**Theorem 17.2** (Conic lower-bound certificates). If $y \in Y^*$ and $s \in K^*$ satisfy $A^*y + s = c$, then $\langle b, y \rangle$ is a lower bound on every primal feasible objective value. The Lagrangian

$$L(x; y, s) := \langle c, x \rangle + \langle y, b - Ax \rangle - \langle s, x \rangle, \qquad s \in K^*,$$

in which the dual-cone slack $s$ is kept explicit, satisfies $\inf_{x \in E} L(x; y, s) = \langle b, y \rangle$ if $A^*y + s = c$, $-\infty$ otherwise.

For primal-feasible $x$ and dual-feasible $(y, s)$:

$$\boxed{\langle c, x \rangle - \langle b, y \rangle \;=\; \langle x, s \rangle \;\geq\; 0.}$$

**The gap is exactly $\langle x, s \rangle$.** This is the object primal-dual IPM maintains. The new feature compared with Lec 5: a single vector slack $s \in K^*$, not a list of scalar multipliers.

# Proof of Theorem 17.2: Conic Weak Duality

**Step 1 (lower bound).** Let $x$ be primal feasible and $A^*y + s = c$. Then

$$\langle c, x\rangle = \langle A^*y + s, x\rangle = \langle y, Ax\rangle + \langle s, x\rangle = \langle y, b\rangle + \langle s, x\rangle \geq \langle b, y\rangle,$$

because $x \in K$ and $s \in K^*$ make $\langle s, x\rangle \geq 0$.

**Step 2 (Lagrangian formula).** Rearrange:

$$L(x; y, s) = \langle b, y\rangle + \langle c - A^*y - s, x\rangle.$$

The infimum over unrestricted $x \in E$ is $\langle b, y\rangle$ if $c - A^*y - s = 0$, else $-\infty$.

**Step 3 (gap identity).** The same calculation with both $x$ and $(y, s)$ feasible gives

$$\langle c, x\rangle - \langle b, y\rangle = \langle s, x\rangle \geq 0.\square$$

# Conic KKT Point

**Definition 17.5** (Conic KKT point). $(x, y, s) \in E \times Y^* \times E^*$ is a KKT point of the conic pair if

$$\boxed{Ax = b,\ x \in K; \quad A^*y + s = c,\ s \in K^*; \quad \langle x, s \rangle = 0.}$$

**Proposition 17.3**. Every conic KKT point certifies primal and dual optimality:

$$\langle c, x \rangle = p^\star = d^\star = \langle b, y \rangle.$$

**Proof.** Every dual-feasible pair gives a lower bound (Thm 17.2). For a KKT point,

$$\langle c, x \rangle - \langle b, y \rangle = \langle x, s \rangle = 0,$$

so the lower bound is tight. □

# Conic Slater

**Theorem 17.4** (Conic Slater). Let $K \subseteq E$ be a closed convex cone, and consider

$$p^\star = \inf\{\langle c, x\rangle : Ax = b,\ x \in K\}.$$

Assume $p^\star \in \mathbb{R}$ and that there exists $\bar{x} \in \operatorname{int}(K)$ with $A\bar{x} = b$. Then the conic dual attains its optimum and

$$\boxed{d^\star = p^\star.}$$

If the primal optimum is attained at $x^\star$, then there exist $y^\star, s^\star$ with $(x^\star, y^\star, s^\star)$ a conic KKT point.

**Form of Slater.** The interior-point assumption $\bar{x} \in \operatorname{int}(K)$ is the cone analog of strict feasibility in Lec 5 / Lec 4 marginal duality.

# Proof of Theorem 17.4: Conic Slater

**Step 1 (set up the perturbation).** On range$(A)$, define

$$p(u) := \inf\{\langle c, x\rangle : Ax = b + u,\ x \in K\}.$$

Choose a right inverse $B : \text{range}(A) \to E$, $ABu = u$. For small $u$, $\bar{x} + Bu \in K$ (since $\bar{x} \in \text{int}\, K$) and $A(\bar{x} + Bu) = b + u$, so $0 \in \text{ri}(\text{dom}\, p)$.

**Step 2 (apply marginal duality, Thm 4.4).** Some $\tilde{y} \in \text{range}(A)^*$ attains the conjugate dual at $u = 0$. Extend to $y \in Y^*$. Then $p(0) = -p^*(\tilde{y})$.

**Step 3 (compute the conjugate).**

$$\begin{aligned} p^*(\tilde{y}) &= \sup_{u \in \text{range}(A)} (\langle \tilde{y}, u\rangle - p(u)) = \sup_{x \in K}(\langle y, Ax - b\rangle - \langle c, x\rangle) \\ &= -\langle b, y\rangle + \sup_{x \in K}\langle A^* y - c, x\rangle. \end{aligned}$$

The sup over $x \in K$ is finite iff $A^* y - c \in K^\circ = -K^*$, i.e., $s := c - A^* y \in K^*$.

**Step 4 (conclude).** In that case $p^*(\tilde{y}) = -\langle b, y\rangle$, so $p^\star = \langle b, y\rangle = d^\star$. If $x^\star$ primal-optimal: gap identity (Thm 17.2) gives $\langle x^\star, s\rangle = 0$, hence KKT. □

# Cones vs. General Set Constraints

**General set constraint.** For closed convex $C$, the equality-constrained problem $\inf\{\langle c, x\rangle : Ax = b,\ x \in C\}$ has dual

$$\sup_y \{\langle b, y\rangle - \sigma_C(A^* y - c)\},$$

where $\sigma_C(z) = \sup_{x\in C}\langle z, x\rangle$ is the support function.

**Cone case $C = K$.** The support function of a cone collapses to the indicator of its polar:

$$\boxed{\sigma_K \;=\; \delta_{K^\circ},} \qquad K^\circ = -K^*,$$

i.e. $\sigma_K(z) = 0$ on $K^\circ$ and $+\infty$ elsewhere. Hence $\sigma_K(A^* y - c) < \infty \iff A^* y + s = c$ for some $s \in K^*$ — exactly the dual-feasibility condition. This is *why cones are algorithmically cleaner than arbitrary convex-set constraints.*

# SOCP Robust Dual Certificate

**Primal (Robust LP / SOCP).** With cone block $(t_i, z_i) := (b_i - a_i^\top x,\ P_i^\top x)$:

$$\min\ c^\top x \quad \text{s.t.} \quad (t_i, z_i) \in \mathcal{Q}^{k_i+1}, \quad i = 1, \dots, m.$$

**Dual.** Let $s_i = (\alpha_i, v_i) \in \mathcal{Q}^{k_i+1}$. The conic dual is

$$\max\ -\sum_{i=1}^{m} \alpha_i b_i \quad \text{s.t.} \quad \sum_{i=1}^{m} (P_i v_i - \alpha_i a_i) = c, \ \ (\alpha_i, v_i) \in \mathcal{Q}^{k_i+1}.$$

**SOC slack.** The second-order cone is *self-dual*, so $\boxed{s_i = (\alpha_i, v_i) \in \mathcal{Q}^{k_i+1}.}$

**Certificate identity.** For every primal-feasible $x$ and dual-feasible $(\alpha, v)$:

$$c^\top x + \sum_{i=1}^{m} \alpha_i b_i \ = \ \sum_{i=1}^{m} \left[\alpha_i t_i + v_i^\top z_i\right] \ = \ \sum_{i=1}^{m} \langle s_i, (t_i, z_i) \rangle \ \geq \ 0.$$

# Max-Cut SDP Dual Certificate

**Primal (Max-Cut SDP).** $\max\{\langle C, X\rangle : X_{ii} = 1,\ X \succeq 0\}$.

**Dual.**

$$\min_{y\in\mathbb{R}^n} \sum_{i=1}^{n} y_i \qquad \text{subject to} \qquad \mathrm{Diag}(y) - C \succeq 0.$$

**PSD slack.** $\boxed{S := \mathrm{Diag}(y) - C \ \succeq\ 0.}$

**Certificate identity.** For every feasible $X$ and $(y, S)$:

$$\sum_{i=1}^{n} y_i - \langle C, X\rangle = \langle \mathrm{Diag}(y) - C, X\rangle = \langle S, X\rangle \geq 0.$$

$S \succeq 0$ is a matrix upper-bound certificate for all legal Gram matrices $X \succeq 0$ with unit diagonal.

# Why the Conic Formulation Keeps the Slack Explicit

The feasible set $X \succeq 0$ can also be written as the scalar nonsmooth inequality $\lambda_{\max}(-X) \le 0$. Same primal feasible matrices, but *different perturbation space*.

**Conic perturbation → matrix slack.** Perturbing $X \succeq 0$ produces a matrix-valued perturbation and a matrix slack $S \succeq 0$.

**Scalar perturbation → subgradient inclusion.** Perturbing $\lambda_{\max}(-X) \le 0$ produces a scalar multiplier $\lambda \ge 0$ and nonsmooth stationarity

$$0 \in C - \mathrm{Diag}(y) - \lambda\, \partial\lambda_{\max}(-X).$$

Unpacking $\partial\lambda_{\max}$ reconstructs $S = \lambda R$ with $R \succeq 0$, $\mathrm{tr}(R) = 1$, supported on the active eigenspace.

**Same bound, different variables.** The conic formulation keeps the matrix slack explicit — exactly what the primal-dual Newton system needs.

# Logarithmically Homogeneous Self-Concordant Barriers

**Definition 17.6** ($\nu$-LHSCB). Let $K \subseteq E$ be a proper cone. A $\nu$-self-concordant barrier $F : \mathrm{int}(K) \to \mathbb{R}$ is *logarithmically homogeneous* (a *$\nu$-LHSCB*) if

$$\boxed{F(\tau x) = F(x) - \nu \log \tau \quad \forall x \in \mathrm{int}(K),\ \forall \tau > 0.}$$

**Canonical examples.** Each $F$ is the log of the cone's defining quantity:

- Nonneg. orthant $K = \mathbb{R}^n_+$: $F(x) = -\sum_{i=1}^n \log x_i$, $\nu = n$.
- Second-order cone $K = \mathcal{Q}^{k+1}$: $F(\tau, z) = -\log(\tau^2 - \|z\|_2^2)$, $\nu = 2$.
- PSD cone $K = \mathbb{S}^n_+$: $F(X) = -\log\det X$, $\nu = n$.

Verify $F(\tau x) = F(x) - \nu \log \tau$ directly. SOC has $\nu = 2$ *independent* of $k$.

# LHSCB Identities

**Proposition 17.5**. Let $F$ be a $\nu$-LHSCB for a proper cone $K$. For every $x \in \mathrm{int}(K)$:

$$\langle \nabla F(x), x \rangle = -\nu, \quad \nabla^2 F(x)\, x = -\nabla F(x), \quad \|\nabla F(x)\|_{x,*} = \sqrt{\nu}.$$

Moreover, $-\nabla F(x) \in \mathrm{int}(K^*)$.

**Proof.** Differentiate $F(\tau x) = F(x) - \nu \log \tau$ in $\tau$ at $\tau = 1$:

$$\langle \nabla F(x), x \rangle = -\nu.$$

Differentiate this identity in $x$: $\nabla^2 F(x)\, x = -\nabla F(x)$.

Therefore

$$\|\nabla F(x)\|_{x,*}^2 = \langle \nabla F(x), (\nabla^2 F(x))^{-1} \nabla F(x) \rangle = \langle \nabla F(x), -x \rangle = \nu.$$

$-\nabla F(x) \in \mathrm{int}(K^*)$ is a standard duality property of LHSCBs (gradient map sends $\mathrm{int}(K) \to -\mathrm{int}(K^*)$). □

# LHSCB Calculus: Derivatives and Combinations

**PSD derivatives.** For $F(X) = -\log\det X$ on $\mathbb{S}^n_{++}$:

$$\nabla F(X) = -X^{-1}, \qquad \nabla^2 F(X)[H] = X^{-1} H X^{-1}.$$

Logarithmic homogeneity: $\langle \nabla F(X), X \rangle = -\operatorname{tr}(X^{-1}X) = -n = -\nu$. ✓

**Combining barriers (Lec 16, Exercises).**

- *Same domain.* If $\Phi_1, \Phi_2$ are $\nu_1$-, $\nu_2$-self-concordant barriers for the *same* $K \subseteq E$, then $\Phi_1 + \Phi_2$ is a $(\nu_1 + \nu_2)$-SCB for $K$.
- *Direct sum / product cone.* If $\Phi_i$ is a $\nu_i$-SCB for $K_i \subseteq E_i$, then $\Phi_1(x_1) + \Phi_2(x_2)$ is a $(\nu_1 + \nu_2)$-SCB for $K_1 \oplus K_2 \subseteq E_1 \oplus E_2$.

# Universal Barriers: Existence for Convex Bodies

**Theorem 17.6**. Let $D \subseteq E$ be compact, convex, with nonempty interior, and let $n = \dim E$. For $x \in \operatorname{int}(D)$, define

$$D_x^\circ := \{s \in E^* : \langle s, y - x \rangle \leq 1 \ \forall y \in D\}.$$

Then

$$F_D(x) := \log \operatorname{vol}(D_x^\circ)$$

is an $n$-self-concordant barrier for $D$; vol is any Lebesgue measure on $E^*$.

*References:* Nesterov–Nemirovskii (1994); Lee–Yue (2021) for sharp $n$-self-concordance.

**Meaning.** Self-concordant barriers are not special accidents of boxes, polytopes, or PSD cones. They exist in dimension-controlled form for arbitrary convex bodies.

**But:** existence is not an implementation recipe. An IPM still needs fast formulas for value, gradient, Hessian, and Newton systems.

# Characteristic Barriers on Cones

**Theorem 17.7** (Güler 1996; see also Nesterov–Nemirovskii 1994). Let $K$ be a proper cone in an $n$-dim space. Define

$$\phi_K(x) := \int_{K^*} e^{-\langle s,x\rangle}\, ds, \qquad F_K(x) := \log \phi_K(x).$$

Then $F_K$ is an $n$-LHSCB for $K$.

**Easy part: logarithmic homogeneity.** Change variables $r = \tau s$:

$$\phi_K(\tau x) = \tau^{-n}\phi_K(x) \quad \Rightarrow \quad F_K(\tau x) = F_K(x) - n \log \tau.$$

**Why this matters here.** The convex-body theorem gives a barrier. The cone theorem gives a barrier compatible with positive scaling, hence with conic central paths and primal-dual slacks. Practical solvers still use the explicit standard barriers from the previous slide.

# Conic Central Path as Perturbed KKT

**Primal barrier subproblem.**

$$x(t) := \operatorname{argmin}\{t\langle c, x\rangle + F(x) : Ax = b,\ x \in \operatorname{int}(K)\}.$$

**Equality-constrained optimality.** Some $\eta(t) \in Y^*$ with

$$tc + \nabla F(x(t)) + A^*\eta(t) = 0.$$

Set $\mu := 1/t$, $y(t) := -\mu\eta(t)$, $s(t) := -\mu\nabla F(x(t))$. Prop 17.5 gives $s(t) \in \operatorname{int}(K^*)$.

**Central-path equations.**

$$\boxed{Ax(t) = b, \quad A^*y(t) + s(t) = c, \quad s(t) = -\mu\nabla F(x(t)).}$$

**Gap on the central path.**

$$\langle x(t), s(t)\rangle = -\mu\langle x(t), \nabla F(x(t))\rangle = \nu\mu.$$

Exact complementarity $\langle x, s\rangle = 0$ is replaced by the controlled perturbation $\langle x(t), s(t)\rangle = \nu\mu$.

# LP Specialization: $x \odot s = \mu \mathbf{1}$

**LP barrier.** $F(x) = -\sum_i \log x_i$ on $\mathbb{R}^n_{++}$, so $\nabla F(x)_i = -1/x_i$.

**Centrality** $s(t) = -\mu \nabla F(x(t))$ **becomes:**

$$s_i(t) = \frac{\mu}{x_i(t)} \iff x_i(t) s_i(t) = \mu.$$

**Coordinatewise product.** Write $(x \odot s)_i := x_i s_i$. The central-path system is

$$\boxed{Ax(t) = b, \quad A^\top y(t) + s(t) = c, \quad x(t) \odot s(t) = \mu \mathbf{1}.}$$

**Interpretation.** Exact LP complementarity $x \odot s = \mathbf{0}$ is replaced by $x \odot s = \mu \mathbf{1}$: each pair $(x_i, s_i)$ has the same nonzero product $\mu$ along the path.

For other standard cones, the rule $s(t) = -\mu \nabla F(x(t))$ gives different explicit formulas (PSD: $S = \mu X^{-1}$; SOC: a formula in $\tau^2 - \|u\|^2$).

# Primal-Dual Newton System (LP)

**Newton in equation-solving form.** For a residual map $R(z)$, solve $DR(z)\Delta z = -R(z)$, then $z^+ = z + \Delta z$.

**Residuals at complementarity level $\mu$.**

$$r_p := b - Ax, \quad r_d := c - A^\top y - s, \quad r_c := \mu\mathbf{1} - x \odot s.$$

**Linearize.** With $X := \mathrm{Diag}(x)$, $S := \mathrm{Diag}(s)$:

$$\boxed{A\Delta x = r_p, \quad A^\top \Delta y + \Delta s = r_d, \quad S\Delta x + X\Delta s = r_c.}$$

**Step length.** Choose $\alpha \in (0, 1]$ so that

$$x + \alpha\Delta x > 0, \quad s + \alpha\Delta s > 0.$$

**Interpretation.** The primal-dual Newton system is Newton's method applied to the residual of perturbed KKT. The same template applies to SOCP/SDP with $\odot$ replaced by the appropriate algebra.

# Primal-Dual Path-Following: One Update

**Algorithm (one LP primal-dual update).** Strictly positive $(x_k, y_k, s_k)$, current $\mu_k$, target $\mu_{k+1} < \mu_k$.

1. Form residuals at the new target:
$$r_p = b - Ax_k, \quad r_d = c - A^\top y_k - s_k, \quad r_c = \mu_{k+1}\mathbf{1} - x_k \odot s_k.$$
2. Solve the Newton system:
$$A\Delta x = r_p, \;\; A^\top \Delta y + \Delta s = r_d, \;\; S_k \Delta x + X_k \Delta s = r_c.$$
3. Choose $\alpha \in (0, 1]$ with $x_k + \alpha\Delta x > 0$ and $s_k + \alpha\Delta s > 0$.
4. Update $(x_{k+1}, y_{k+1}, s_{k+1}) = (x_k, y_k, s_k) + \alpha(\Delta x, \Delta y, \Delta s)$.

**This is a one-step template.** A rigorous short-step method defines a central neighborhood and proves, by induction, that the update maps $\mu_k$-neighborhood $\rightarrow$ $\mu_{k+1}$-neighborhood while preserving positivity.

Standard analyses for LP and the standard conic families: Nesterov-Nemirovskii (1994), Wright (1997), Renegar (2001). Modern solvers use predictor-corrector / infeasible-start variants of the same perturbed-KKT Newton idea.

# Symmetric Cones and Self-Scaled Barriers

**Centrality $s = -\mu \nabla F(x)$ for the three standard cones:**

| Cone | Barrier | Centrality |
|---|---|---|
| $\mathbb{R}^n_+$ | $-\sum_i \log x_i$ | $s_i = \mu / x_i$, i.e., $x_i s_i = \mu$ for all $i$. |
| $\mathcal{Q}^{d+1}$ | $-\log(\tau^2 - \|u\|_2^2)$ | For $x = (\tau, u)$, $d := \tau^2 - \|u\|_2^2$: $s = (2\mu\tau/d, -2\mu u/d)$. |
| $\mathbb{S}^n_+$ | $-\log\det X$ | $S = \mu X^{-1}$, i.e., $X^{1/2} S X^{1/2} = \mu I$. |

**What makes LP, SOCP, SDP special?** Two shared structural features:

- *Self-dual.* $K^* = K$ under the inner product, so the dual slack $s$ lives in the *same* cone as $x$.
- *Closed-form "product $= \mu \cdot$ identity".* The centrality equation $s = -\mu \nabla F(x)$ becomes

$$x_i s_i = \mu \text{ (LP)}, \quad \text{Jordan product} = \mu e \text{ (SOCP)}, \quad X^{1/2} S X^{1/2} = \mu I \text{ (SDP)}.$$

This shared algebra is the basis of Nesterov–Todd scaling (1997), which makes the primal-dual Newton system symmetric across all three cones.

# Summary

**Conic form.** Nonlinear convex constraint = membership in a cone after a linear map (SOCP from robust LP, SDP from quadratic binary).

**Conic duality.** Dual slack $s \in K^*$ instead of scalar multipliers; gap $= \langle x, s \rangle \geq 0$; conic Slater (interior-point primal) $\Rightarrow$ strong duality + KKT existence.

**LHSCB.** Logarithmically-homogeneous self-concordant barrier on a proper cone:

$$F(\tau x) = F(x) - \nu \log \tau \;\Rightarrow\; \langle \nabla F(x), x \rangle = -\nu,\; -\nabla F(x) \in \mathrm{int}(K^*).$$

**Conic central path.** $s(t) = -\mu \nabla F(x(t))$, $\langle x(t), s(t) \rangle = \nu\mu$. Perturbed KKT.

**Primal-dual Newton system (LP).** $A\Delta x = r_p$, $A^\top \Delta y + \Delta s = r_d$, $S\Delta x + X\Delta s = r_c$. Step length keeps $x, s > 0$.

**Symmetric cones.** LP, SOCP, SDP are self-dual + homogeneous; enough algebra for clean primal-dual scaling (Nesterov-Todd).

# Bibliographic Notes


- **Yu. Nesterov & A. Nemirovskii, Interior-Point Polynomial Algorithms in Convex Programming, SIAM (1994).** Self-concordant barriers, universal barrier, primal-dual algorithms.
- **S. Wright, Primal-Dual Interior-Point Methods, SIAM (1997).** Detailed LP primal-dual analysis.
- **J. Renegar, A Mathematical View of Interior-Point Methods, MPS-SIAM (2001).** Modern self-concordant viewpoint.
- **Yu. Nesterov & M. J. Todd, "Self-scaled barriers and interior-point methods," Math. Oper. Res. 22 (1997).** Self-scaled barriers + symmetric-cone primal-dual scaling.
- **O. Güler (1996).** Logarithm of characteristic function as LHSCB.
- **Y. T. Lee & M. Yue (2021).** Sharp $n$-self-concordance of the universal barrier.
- **M. X. Goemans & D. P. Williamson (1995).** Max-Cut SDP + rounding.

# Final Exam and Other Logistics

**Final exam.** Monday **May 26**, **1:00–4:00 pm**.

- One A4 cheat sheet, **double-sided**, allowed.
- **Must be handwritten** — no printed/typeset sheets.
- *Some HW problems will appear in the final.*

**Practice exam.** Released **May 22**, with solutions.

**Bonus problems.** Deadline **AoE, May 22**.

**In-class bonus.** If I promised you bonus points in lecture, please email me a reminder (with the lecture, if you remember it, and the points) so I do not miss it.

**Extra office hours.** Beining will hold an additional OH **next Monday**.