Lecture 3: Linear Programming

TTIC 31070 / CMSC 35470 / BUSF 36903 / STAT 31015

Convex Optimization

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Linear Programming

Lecture 2 built the separation toolkit. Lecture 3 applies it to the simplest optimization model — **linear programming (LP)**.

Example: A Graduate-Student Workload LP

Example 3.1. A graduate student allocates x_1 hours to research-focused work and x_2 hours to course-focused work.

	Research progress (units/hr)	Course preparation (units/hr)	Strain cost (per hr)
Research-focused (x_1 hrs)	2	1	5
Course-focused (x_2 hrs)	1	2	4
Total required	≥ 6	≥ 9	

From Table to LP

Objective: minimize total strain $5x_1 + 4x_2$.

Constraints:

- Research: $2x_1 + x_2 \geq 6$
- Course: $x_1 + 2x_2 \geq 9$
- Hours are nonneg: $x_1, x_2 \geq 0$

$$
\begin{aligned}
\min_{x_1, x_2} \quad & 5x_1 + 4x_2 \\
\text{s.t.} \quad & 2x_1 + x_2 \geq 6 \\
& x_1 + 2x_2 \geq 9 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

The constraint coefficients, the RHS, and the objective all come directly from the table.

Optimality and Certificate

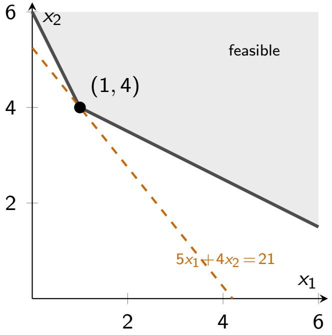
Optimal solution: $(x_1, x_2) = (1, 4)$ with value 21.

Question. How do I convince my peer student that this is truly optimal?

Certificate: $2(2x_1 + x_2 \geq 6) + (x_1 + 2x_2 \geq 9)$ gives $5x_1 + 4x_2 \geq 21$.

So every feasible point has value ≥ 21.

The weights $(2, 1)$ are already hinting at something dual.



Canonical Primal and Dual LP

Definition 3.1 (Canonical LP). Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$.

Primal: $\inf_{x \geq 0} \{c^\top x : \ Ax \geq b\}$

Dual: $\sup_{y \geq 0} \{b^\top y : \ A^\top y \leq c\}$

Weak Duality

Every dual-feasible y gives a lower bound on every primal-feasible x.

Lemma 3.1 (Weak duality for LP). If $Ax \geq b$, $x \geq 0$, $A^\top y \leq c$, $y \geq 0$, then

$$b^\top y \leq c^\top x.$$

Proof. $b^\top y \leq (Ax)^\top y = x^\top (A^\top y) \leq x^\top c = c^\top x$. □

In the workload example: dual weights $(2, 1)$ are feasible and give $b^\top y = 2 \cdot 6 + 1 \cdot 9 = 21 = c^\top x^\star$.

Strong Duality for LP

Weak duality says every dual point gives a lower bound. The natural question: **how tight can that bound be?**

Theorem 3.2 (Strong duality). If the primal LP is feasible with finite optimal value, then the dual is also feasible, attains its optimum, and

$$\inf\{c^\top x : Ax \geq b,\ x \geq 0\} = \sup\{b^\top y : A^\top y \leq c,\ y \geq 0\}.$$

For linear programs, the best dual certificate is **exactly tight**. The proof uses the next theorem as its engine.

Farkas' Lemma: the Infeasibility Certificate

The proof of strong duality needs an infeasibility certificate.

Theorem 3.3 (Farkas' lemma). Exactly one of the following holds:

1. $\exists\, x \geq 0$ with $Ax \geq b$ (feasible);
2. $\exists\, y \geq 0$ with $A^\top y \leq 0$ and $b^\top y > 0$ (infeasibility certificate).

Proof that (2) $\Rightarrow$ not (1) (easy direction).

If (2) holds and $x \geq 0$ with $Ax \geq b$, then $b^\top y \leq y^\top Ax = x^\top A^\top y \leq 0 < b^\top y$. $\square$

Farkas' Lemma: Proof of not (1) $\Rightarrow$ (2)

Proof that not (1) $\Rightarrow$ (2) (hard direction).

Define the cone $K := -A\mathbb{R}^n_+ + \mathbb{R}^m_+ = \{-Ax + s : x \geq 0,\ s \geq 0\}$.

K is a closed convex cone (image of $\mathbb{R}^n_+ \times \mathbb{R}^m_+$ under a linear map).

Not (1) means $-b \notin K$.

Apply cone separation (Theorem 2.7): $\exists\, y$ with

$$\langle y, z\rangle \geq 0 \ \ \forall z \in K, \qquad \langle y, -b\rangle < 0.$$

- $e_i \in \mathbb{R}^m_+ \subseteq K \Rightarrow y_i \geq 0$
- $-Ae_j \in K \Rightarrow (A^\top y)_j \leq 0$
- $\langle y, -b\rangle < 0 \Rightarrow b^\top y > 0$

This is exactly statement (2). □

Recall: Strong Duality

Theorem 3.2 (Strong duality). If the primal LP is feasible with finite optimal value, then

$$\inf_{x \geq 0} \{c^\top x : Ax \geq b\} = \sup_{y \geq 0} \{b^\top y : A^\top y \leq c\}$$

and the dual optimum is attained.

We now prove this using Farkas.

Proof of Strong Duality (1/2)

Fix $\mu < p^\star$. The augmented system

$$\begin{pmatrix} A \\ -c^\top \end{pmatrix} x \geq \begin{pmatrix} b \\ -\mu \end{pmatrix}, \qquad x \geq 0$$

is infeasible. By Farkas, $\exists\,(y, \alpha) \geq 0$ with

$$A^\top y - \alpha c \leq 0, \qquad b^\top y - \alpha\mu > 0.$$

Claim: $\alpha > 0$.

If $\alpha = 0$: then $A^\top y \leq 0$, $y \geq 0$, $b^\top y > 0$. But by Farkas again, this would certify that $Ax \geq b$, $x \geq 0$ is infeasible — contradicting our hypothesis that the problem is feasible.

Proof of Strong Duality (2/2)

Since $\alpha > 0$, set $\bar{y} := y/\alpha$. Then:

- $A^\top \bar{y} \leq c$ and $\bar{y} \geq 0$ (dual feasible)
- $b^\top \bar{y} = (b^\top y)/\alpha > \mu$ (dual value $> \mu$)

This works for every $\mu < p^\star$, so $d^\star \geq p^\star$.

Weak duality gives $d^\star \leq p^\star$.

Therefore $p^\star = d^\star$.

Finally, let $\mu_k \uparrow p^\star$. The dual-feasible points $\bar{y}_k$ live in a bounded polyhedron with $b^\top \bar{y}_k > \mu_k$. A compactness argument yields an optimal dual solution $y^\star$ with $b^\top y^\star = p^\star$. $\square$

Complementary Slackness

Strong duality tells us the optimal values match. But **which constraints are active** at optimality?

Theorem 3.4 (Complementary slackness). $x^\star$ primal optimal and $y^\star$ dual optimal iff both are feasible and:

$$y_i^\star\big((Ax^\star)_i - b_i\big) = 0 \quad \forall\, i, \qquad x_j^\star\big(c_j - (A^\top y^\star)_j\big) = 0 \quad \forall\, j.$$

Proof idea. Expand the duality gap: $c^\top x^\star - b^\top y^\star = \sum_i y_i^\star((Ax^\star)_i - b_i) + \sum_j x_j^\star(c_j - (A^\top y^\star)_j)$. All terms ≥ 0; zero gap forces each term to vanish. $\square$

At optimality, every positive dual multiplier sits on a tight primal constraint, and vice versa.

Value Function

Weak duality gives a lower bound on the optimal value of one LP with right-hand side b. But one can understand the dual variable much better by enlarging the picture: instead of fixing b, ask how the optimal cost changes when the requirement vector is **perturbed**.

Definition 3.2 (Value function). Fix A, c. The requirement-perturbation value function is

$$V(u) := \inf\{c^\top x : Ax \geq u,\ x \geq 0\}.$$

The Value Function Is Convex

$$V(u) := \inf\{c^\top x : Ax \geq u,\ x \geq 0\}.$$

Lemma 3.5. V is convex.

Proof (direct). Average feasible points for u^1 and u^2: $\bar{x} = \theta x^1 + (1-\theta)x^2$ is feasible for $\theta u^1 + (1-\theta)u^2$ with $c^\top \bar{x} \leq \theta c^\top x^1 + (1-\theta)c^\top x^2$. □

Geometric view. Taking $\inf_x$ is projecting out x from $\mathrm{epi}(\Phi)$: projection preserves convexity, so V is convex.

This "partial inf = projection of epigraph" viewpoint is exactly what Lecture 4 abstracts into $p(u) = \inf_x \Phi(x, u)$.

Shadow Prices = Supports of the Value Function

Instead of fixing b, ask: how does the optimal cost change when the requirement vector is perturbed?

Every dual-feasible y gives an affine lower bound on the **entire** value function:

$$V(u) \geq u^\top y \qquad \forall\, u \in \mathbb{R}^m.$$

If strong duality holds at b with dual optimum $y^\star$: $V(b) = b^\top y^\star$.
So $y^\star$ is a **supporting covector** of V at b.

If V is differentiable at b: $y^\star = \nabla V(b)$.

The component $y_i^\star$ = marginal increase in optimal cost per unit increase in the i-th requirement.

At kinks, replace the gradient by a subgradient. Lecture 4 abstracts this into $p(u) = \inf_x \Phi(x, u)$.

Example: Workload Value Function

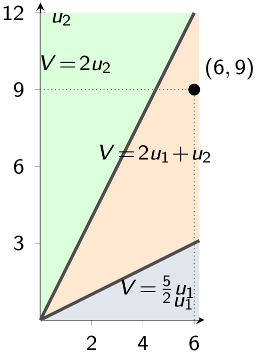
Example 3.2. Variable requirements (u_1, u_2):

$$V(u_1, u_2) = \max\{0,\ \tfrac{5}{2}u_1,\ 2u_1{+}u_2,\ 2u_2\}.$$

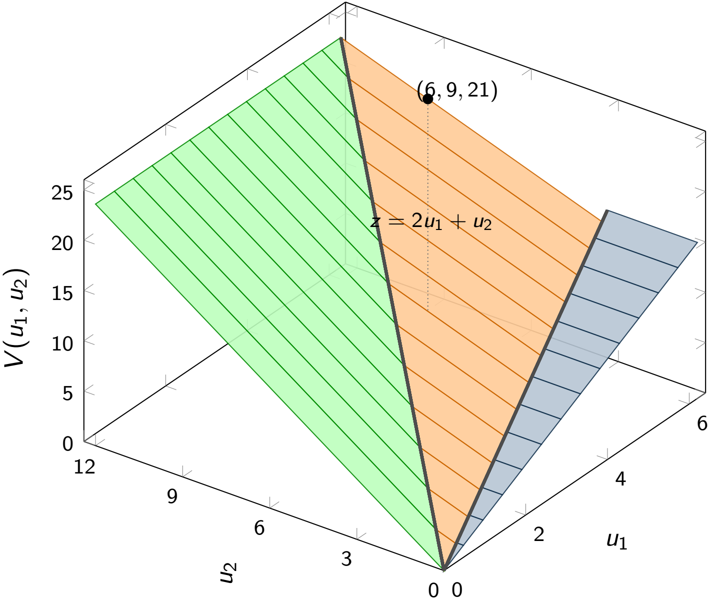
At $(6, 9)$: active piece $2u_1 + u_2$, $V(6, 9) = 21$.

$\nabla V(6, 9) = (2, 1) =$ dual optimum.

Increasing research req. by 1 unit $\Rightarrow$ cost $\uparrow$ by ≈ 2.
Increasing course req. by 1 unit $\Rightarrow$ cost $\uparrow$ by ≈ 1.



The Value Function Is Piecewise Linear



Summary & What's Next

Today:

- Canonical primal and dual LP
- Weak duality: every dual point gives a lower bound
- Farkas' lemma: infeasibility certificate from cone separation
- Strong duality: the best certificate is tight
- Complementary slackness: optimality = matching active constraints
- Value function and shadow prices as subgradients

Next lecture:

- Fenchel conjugates and Fenchel–Moreau
- The perturbation/marginal viewpoint for general convex duality
- From LP shadow prices to the marginal duality theorem