Lecture 16: Self-Concordant Barriers and Primal Interior-Point Methods

Lecture 15 explained why Newton's method is stable for self-concordant functions. Lecture 16 is the first interior-point method (IPM) lecture: we apply the self-concordant Newton calculus to constrained optimization by keeping all iterates inside the feasible region and enforcing the boundary through a barrier. The central path is the curve tracked by this feasible-start primal IPM. The point of this lecture is deliberately not conic modeling. We work with a general closed convex feasible set K and a self-concordant barrier on $\operatorname{int}(K)$. Cones and primal-dual implementations are postponed to Lecture 17.

Throughout this lecture, E denotes a finite-dimensional real vector space, $K \subseteq E$ is a closed convex set with nonempty interior, and $c \in E^*$. The basic feasible-start problem is

$$p^\star := \inf \{\langle c, x \rangle : x \in K\}.$$

A point $x \in \operatorname{int}(K)$ is called strictly feasible. The algorithms in this lecture assume that a strictly feasible point is already available; boundary-feasible and unknown-feasibility starts are not treated here.

16.1 Self-Concordant Barriers and the Barrier Parameter

A barrier is a function defined on $\operatorname{int}(K)$ that blows up at the boundary. A self-concordant barrier is a barrier whose Hessian geometry is stable in the sense of Lecture 15.

Definition 16.1 (ν-self-concordant barrier)**.** A function $\Phi : \operatorname{int}(K) \to \mathbb{R}$ is called a ν-self-concordant barrier for K if

1. Φ is self-concordant on $\operatorname{int}(K)$ in the sense of Definition 15.1;
2. $\nabla^2\Phi(x) \succ 0$ for every $x \in \operatorname{int}(K)$;
3. Φ blows up at the boundary:
$$x_j \in \operatorname{int}(K), \quad x_j \to \bar{x} \in \partial K \quad \Longrightarrow \quad \Phi(x_j) \to +\infty;$$
4. for every $x \in \operatorname{int}(K)$,
$$\|\nabla\Phi(x)\|_{x,*} \le \sqrt{\nu}.$$

Example 16.1 (Standard barriers)**.** The nonnegative orthant $\mathbb{R}^m_+$ has the logarithmic barrier

$$\Phi(x) := -\sum_{i=1}^m \log x_i, \qquad x \in \mathbb{R}^m_{++},$$

with parameter $\nu = m$. The positive-semidefinite cone $\mathbb{S}^n_+$ has the log-determinant barrier

$$\Phi(X) := -\log\det X, \qquad X \succ 0,$$

with parameter $\nu = n$. These two examples are the prototypes behind LP and SDP interior-point methods.

Lemma 16.1 (Dikin ellipsoid inclusion)**.** *Let Φ be a self-concordant barrier for K. For every $x \in \operatorname{int}(K)$,*

$$\{x + u : \|u\|_x < 1\} \subseteq \operatorname{int}(K).$$

Proof of Lemma 16.1. Fix $x \in \operatorname{int}(K)$ and $u \in E$ with $r := \|u\|_x < 1$. Suppose for contradiction that $x + u \notin \operatorname{int}(K)$. Since $\operatorname{int}(K)$ is open and convex, the segment $x + su$ first hits the boundary at some $\tau \in (0, 1]$: for $s \in [0, \tau)$, $x + su \in \operatorname{int}(K)$, and $x + \tau u \in \partial K$.
For $s < \tau$, the one-dimensional self-concordant upper bound from Theorem 15.6 gives

$$\Phi(x + su) \leq \Phi(x) + s\,\langle \nabla\Phi(x), u\rangle + \omega^*(sr).$$

Because $\tau r < 1$, the right-hand side remains bounded above as $s \uparrow \tau$. This contradicts the boundary blow-up of Φ at $x + \tau u$. Hence $x + u \in \operatorname{int}(K)$. $\square$

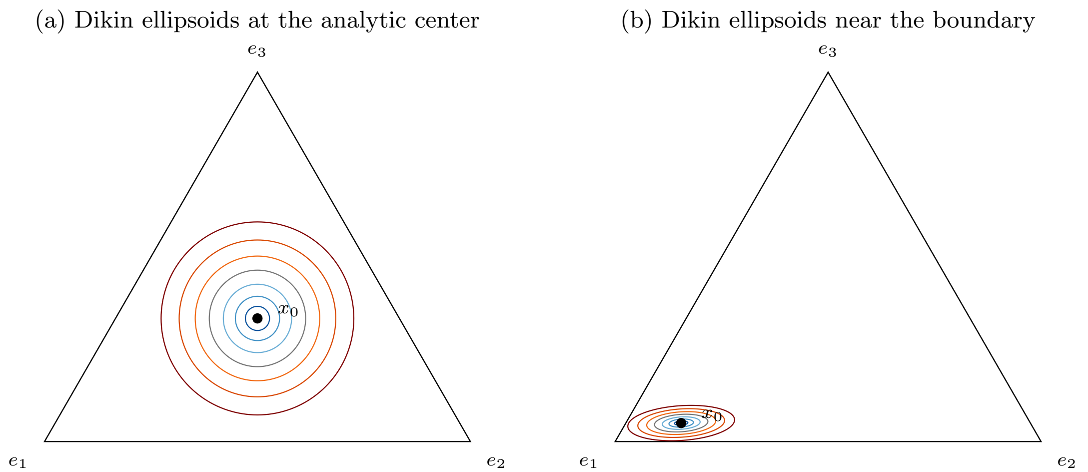

Figure 1: Dikin ellipsoids for the logarithmic barrier $\Phi(x) = -\sum_{i=1}^3 \log x_i$ on $\Delta_3 = \left\{x \in \mathbb{R}^3_+ : \sum_{i=1}^3 x_i = 1\right\}$.[1] The curves are local Hessian balls $\left\{y : \|y - x_0\|_{x_0} \leq r\right\}$ for radii $r < 1$.

Compare Figure 1 with the entropy mirror-descent contours in Figure 8.3: both geometries become anisotropic near the boundary, but the logarithmic barrier is strong enough that its local unit balls stay inside the simplex by Lemma 16.1.

[1] Here Δ_3 is viewed in its affine hull, equivalently after translating to the two-dimensional linear space $\left\{h \in \mathbb{R}^3 : \sum_i h_i = 0\right\}$.

Lemma 16.2 (Global barrier inequality). *Let Φ be a ν-self-concordant barrier for K. Then*

$$\langle \nabla \Phi(x), y - x\rangle \leq \nu \qquad \forall x \in \operatorname{int}(K),\ \forall y \in K.$$

Proof of Lemma 16.2. Fix $x \in \operatorname{int}(K)$ and $y \in K$. Let

$$\phi(t) := \Phi(x + t(y - x)), \qquad t \in [0, 1).$$

If $\phi'(0) \leq 0$, then the desired inequality is immediate. Assume $\phi'(0) > 0$. Since ϕ is convex, ϕ' is nondecreasing, and therefore $\phi'(t) \geq \phi'(0) > 0$ for every $t \in [0, 1)$.
For $t \in [0, 1)$, the local gradient bound in Definition 16.1 gives

$$|\phi'(t)| = |\langle \nabla \Phi(x + t(y - x)), y - x\rangle| \leq \sqrt{\nu}\sqrt{\phi''(t)}.$$

Since $\phi'(t) > 0$, this implies, for every $t \in [0, 1)$,

$$\phi''(t) \geq \frac{\phi'(t)^2}{\nu}.$$

We may divide by $\phi'(t)^2$ and get

$$\frac{d}{dt}\frac{1}{\phi'(t)} = -\frac{\phi''(t)}{\phi'(t)^2} \leq -\frac{1}{\nu}.$$

Integration gives

$$0 \leq \frac{1}{\phi'(t)} \leq \frac{1}{\phi'(0)} - \frac{t}{\nu}, \forall t \in [0, 1).$$

Letting $t \uparrow 1$ gives $\phi'(0) \leq \nu$, which is exactly

$$\langle \nabla \Phi(x), y - x\rangle \leq \nu.$$

□

16.2 Barrier Subproblems and the Central Path

Let Φ be a ν-self-concordant barrier for K. For $t > 0$, define the barrier objective on $\operatorname{int}(K)$ by

$$F_t(x) := t\ \langle c, x\rangle + \Phi(x).$$

The central path is the family of points

$$x(t) := \arg\min_{x \in \operatorname{int}(K)} F_t(x).$$

Here uniqueness follows from $\nabla^2 \Phi(x) \succ 0$, so F_t is strictly convex. For simplicity, throughout this lecture we assume each F_t attains its minimum.

The Newton decrement of F_t at an interior point $x \in \operatorname{int}(K)$ is

$$\lambda_{F_t}(x) := \|\nabla F_t(x)\|_{x,*} = \sqrt{\langle \nabla F_t(x), (\nabla^2 \Phi(x))^{-1} \nabla F_t(x)\rangle}.$$

Here $\nabla^2 F_t(x) = \nabla^2 \Phi(x)$ because the objective $\langle c, x \rangle$ is linear. The exact central-path condition is

$$\lambda_{F_t}(x(t)) = 0,$$

or equivalently

$$tc + \nabla\Phi(x(t)) = 0.$$

Theorem 16.3 (Barrier gap)**.** *Let $x(t)$ be the central-path point at parameter $t > 0$. Then*

$$\langle c, x(t) \rangle - p^\star \le \frac{\nu}{t}.$$

Proof of Theorem 16.3. Fix any feasible $y \in K$. Centrality gives

$$tc + \nabla\Phi(x(t)) = 0.$$

Hence

$$t(\langle c, x(t) \rangle - \langle c, y \rangle) = \langle \nabla\Phi(x(t)), y - x(t) \rangle .$$

By Lemma 16.2, the right-hand side is at most ν. Thus

$$\langle c, x(t) \rangle - \langle c, y \rangle \le \frac{\nu}{t}.$$

Taking the infimum over $y \in K$ gives the claim. □

Remark 16.1 (Nonlinear objectives)*.* The barrier-gap argument extends to a differentiable convex objective f_0: if $x(t)$ minimizes $tf_0 + \Phi$ over $\mathrm{int}(K)$, then convexity of f_0 and Lemma 16.2 give

$$f_0(x(t)) - p^\star \le \frac{\nu}{t}.$$

The short-step analysis below is cleaner for linear objectives because $\nabla^2(t \langle c, x \rangle + \Phi) = \nabla^2\Phi$ does not change with t. For general self-concordant objectives, the barrier objective $tf_0 + \Phi$ can still be treated by self-concordant minimization theory, but the change-of-t estimates require extra bookkeeping.

16.3 A Short-Step Primal Interior-Point Method

The path-following algorithm keeps the Newton decrement $\lambda_{F_{t_k}}(x_k)$ small, then increases t_k slightly and recenters by one Newton step. Points with small decrement are often called a Newton neighborhood of the central path. The two elementary mechanisms below are: increasing t enlarges the decrement by $O(\sqrt{\nu})$, while one full Newton step maps a small decrement quadratically closer to zero.

Algorithm 1 Short-step primal interior-point method under strict feasibility

Require: Numerical constant $0 < \delta \le 1/10$ (for instance $\delta = 1/10$), strictly feasible $x_0 \in \operatorname{int}(K)$, parameter $t_0 > 0$ with $\lambda_{F_{t_0}}(x_0) \le \delta$, target accuracy $\varepsilon > 0$.

for $k = 0, 1, 2, \dots$ **do**
  if $t_k \ge 2\nu/\varepsilon$ **then**
    return x_k.
  end if
  Set $t_{k+1} = t_k(1 + \delta/\sqrt{\nu})$.
  Compute the Newton direction

$$d_k = -(\nabla^2\Phi(x_k))^{-1}\big(t_{k+1}c + \nabla\Phi(x_k)\big).$$

  Set $x_{k+1} = x_k + d_k$.
end for

Algorithm 1 is an interior-point method in the literal sense: the iterates remain strictly feasible, and the boundary of K is handled by the barrier rather than by projection. This is the clean primal path-following mechanism in its simplest form. Lecture 17 explains how conic solvers reintroduce equality constraints and implement the same central-path idea using primal-dual variables and perturbed KKT equations.

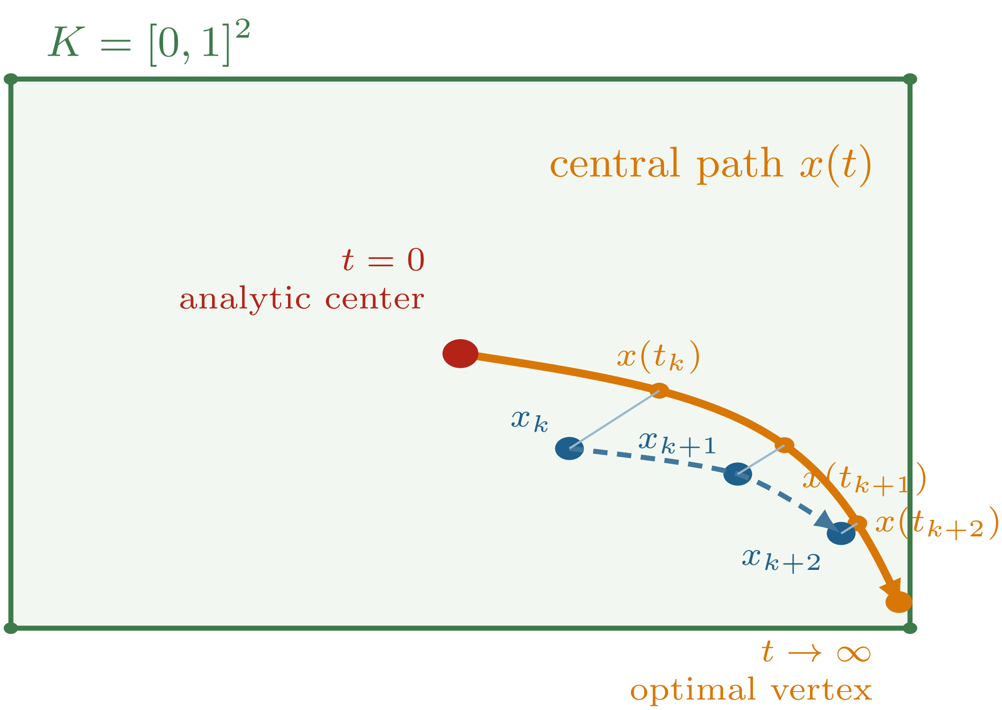

Figure 2: Central-path tracking on the two-dimensional Boolean cube $K = [0,1]^2$. The orange curve is the exact central path for the logarithmic barrier on the box and the linear objective $\langle c, x\rangle = -x_1 + \frac{1}{4}x_2$. The blue iterates x_k, x_{k+1}, x_{k+2} are schematic: they form a separate algorithmic path whose distance to the corresponding central point decreases after each re-centering step.

Lemma 16.4 (Changing t changes the decrement by $O(\sqrt{\nu})$)**.** *Let* $x \in \text{int}(K)$*. Let*

$$t^+ = (1+\eta)t$$

with $\eta > 0$*. Then*

$$\lambda_{F_{t^+}}(x) \le (1+\eta)\lambda_{F_t}(x) + \eta\sqrt{\nu}.$$

Proof of Lemma 16.4. At a fixed point x, all objectives $F_s(z) = s\,\langle c, z\rangle + \Phi(z)$ have the same Hessian:

$$\nabla^2 F_s(x) = \nabla^2 \Phi(x).$$

Thus the local norm and dual local norm used to measure $\lambda_{F_s}(x)$ do not depend on s. Moreover,

$$\lambda_{F_s}(x) = \|\nabla F_s(x)\|_{x,*}\,, \qquad \|\nabla \Phi(x)\|_{x,*} \le \sqrt{\nu}$$

by Definition 16.1.
We have

$$\nabla F_{t^+}(x) = \nabla F_t(x) + \eta t\, c.$$

Also,

$$tc = \nabla F_t(x) - \nabla \Phi(x).$$

Taking the dual local norm,

$$\|tc\|_{x,*} \le \|\nabla F_t(x)\|_{x,*} + \|\nabla \Phi(x)\|_{x,*} \le \lambda_{F_t}(x) + \sqrt{\nu}.$$

Therefore

$$\lambda_{F_{t^+}}(x) \le \lambda_{F_t}(x) + \eta\,\|tc\|_{x,*} \le \lambda_{F_t}(x) + \eta(\lambda_{F_t}(x) + \sqrt{\nu}) = (1+\eta)\lambda_{F_t}(x) + \eta\sqrt{\nu}.$$

□

Remark 16.2 (Two roles of the barrier parameter). The constant 2 in the definition of self-concordance is a normalization. The parameter ν in Definition 16.1 is different: it measures the size of the barrier. Its two algorithmic effects are:

$$\sqrt{\nu} \quad \text{controls short-step path stability,}$$

and

$$\frac{\nu}{t} \quad \text{controls central-path objective accuracy.}$$

Lemma 16.5 (One Newton step recenters)**.** *Let* $t > 0$*, and let* $x \in \text{int}(K)$ *satisfy*

$$\lambda_{F_t}(x) < 1.$$

Let x^+ *be the full Newton step for minimizing* F_t*. Then*

$$x^+ \in \text{int}(K), \qquad \lambda_{F_t}(x^+) \le \left(\frac{\lambda_{F_t}(x)}{1 - \lambda_{F_t}(x)}\right)^2.$$

Proof of Lemma 16.5. The function $F_t = t\langle c, x\rangle + \Phi(x)$ is standard self-concordant by Proposition 15.2. Since $\|x^+ - x\|_x = \lambda_{F_t}(x) < 1$, the Dikin ellipsoid inclusion Lemma 16.1 guarantees that $x^+ \in \operatorname{int}(K)$. The full-step decrement recursion Theorem 15.9 gives the displayed bound. □

Theorem 16.6 (Explicit short-step schedule)**.** *Assume $\nu \geq 1$, and fix a numerical constant*

$$0 < \delta \leq \frac{1}{10}.$$

Let $t_0 > 0$ and suppose

$$\lambda_{F_{t_0}}(x_0) \leq \delta.$$

Define

$$t_{k+1} := t_k \left(1 + \frac{\delta}{\sqrt{\nu}}\right).$$

At iteration k, first increase t_k to t_{k+1}, then take one full Newton step for $F_{t_{k+1}}$. Then

$$\lambda_{F_{t_k}}(x_k) \leq \delta \qquad \forall k \geq 0.$$

Moreover, the output x_N satisfies

$$\langle c, x_N\rangle - p^\star \leq \frac{2\nu}{t_N}.$$

In particular, $t_N \geq 2\nu/\varepsilon$ gives an ε-optimal output.

Proof of Theorem 16.6. **Step 1: the decrement invariant.** Assume $\lambda_{F_{t_k}}(x_k) \leq \delta$. By Lemma 16.4,

$$\lambda_{F_{t_{k+1}}}(x_k) \leq \left(1 + \frac{\delta}{\sqrt{\nu}}\right)\delta + \delta \leq \delta + \frac{\delta^2}{\sqrt{\nu}} + \delta \leq \delta_{\text{mid}}, \qquad \delta_{\text{mid}} := 2\delta + \delta^2.$$

Thus increasing t moves the point from the δ-neighborhood into a larger δ_{mid}-neighborhood. Since $0 < \delta \leq 1/10$, we have $\delta_{\text{mid}} < 1$, and

$$\frac{\delta_{\text{mid}}}{1 - \delta_{\text{mid}}} = \frac{\delta(2+\delta)}{1 - 2\delta - \delta^2} \leq \frac{210}{79}\delta.$$

Hence

$$\left(\frac{\delta_{\text{mid}}}{1 - \delta_{\text{mid}}}\right)^2 \leq \left(\frac{210}{79}\right)^2 \delta^2 \leq \delta,$$

where the last inequality uses $\delta \leq 1/10 < (79/210)^2$. Therefore one full Newton step for $F_{t_{k+1}}$ and Lemma 16.5 give

$$\lambda_{F_{t_{k+1}}}(x_{k+1}) \leq \left(\frac{\lambda_{F_{t_{k+1}}}(x_k)}{1 - \lambda_{F_{t_{k+1}}}(x_k)}\right)^2 \leq \left(\frac{\delta_{\text{mid}}}{1 - \delta_{\text{mid}}}\right)^2 \leq \delta.$$

This proves the decrement invariant by induction.

Step 2: distance to the exact central point. Let $x(t_N)$ be the exact central point at parameter t_N. The invariant gives $\lambda_{F_{t_N}}(x_N) \leq \delta$. The decrement certificate Theorem 15.8 therefore gives

$$F_{t_N}(x_N) - F_{t_N}(x(t_N)) \leq \omega^*(\delta).$$

On the other hand, applying the lower inequality in Theorem 15.6 at the minimizer $x(t_N)$, where $\nabla F_{t_N}(x(t_N)) = 0$, gives

$$F_{t_N}(x_N) - F_{t_N}(x(t_N)) \geq \omega(\|x_N - x(t_N)\|_{x(t_N)}).$$

Combining the last two displays yields

$$\omega(\|x_N - x(t_N)\|_{x(t_N)}) \leq \omega^*(\delta).$$

Since ω is increasing on $[0, \infty)$,

$$\|x_N - x(t_N)\|_{x(t_N)} \leq \rho_\delta, \qquad \rho_\delta := \omega^{-1}(\omega^*(\delta)).$$

We will use the following elementary bound on ρ_δ:

$$\rho_\delta \leq \frac{\delta}{1-\delta}.$$

Indeed, since ω is increasing on $[0, \infty)$, it suffices to prove

$$\omega\left(\frac{\delta}{1-\delta}\right) \geq \omega^*(\delta).$$

Using

$$\omega(r) = r - \log(1+r), \qquad \omega^*(\delta) = -\delta - \log(1-\delta),$$

the difference is

$$\omega\left(\frac{\delta}{1-\delta}\right) - \omega^*(\delta) = \frac{\delta}{1-\delta} + \delta + 2\log(1-\delta).$$

This function is 0 at $\delta = 0$, and its derivative is

$$\frac{1}{(1-\delta)^2} + 1 - \frac{2}{1-\delta} = \frac{\delta^2}{(1-\delta)^2} \geq 0.$$

Hence the difference is nonnegative for $0 \leq \delta < 1$, proving the bound.
Step 3: objective accuracy. By Theorem 16.3,

$$\langle c, x(t_N)\rangle - p^\star \leq \frac{\nu}{t_N}.$$

Since centrality gives

$$t_N c = -\nabla\Phi(x(t_N)),$$

we have

$$t_N(\langle c, x_N\rangle - \langle c, x(t_N)\rangle) \leq \|\nabla\Phi(x(t_N))\|_{x(t_N),*}\,\|x_N - x(t_N)\|_{x(t_N)} \leq \sqrt{\nu}\frac{\delta}{1-\delta}.$$

Adding the last two estimates gives

$$\langle c, x_N\rangle - p^\star \leq \frac{\nu}{t_N} + \frac{\sqrt{\nu}}{t_N}\frac{\delta}{1-\delta} \leq \frac{2\nu}{t_N},$$

because $\nu \geq 1$ and $\delta \leq 1/10$ imply $\delta/(1-\delta) \leq 1/9 \leq 1$. □

Corollary 16.7 (Iteration complexity)**.** *Under the hypotheses of Theorem 16.6, the number of Newton steps needed to reach* $t_N \geq 2\nu/\varepsilon$ *is*

$$O\Big(\sqrt{\nu}\log\frac{\nu}{\varepsilon t_0}\Big).$$

More explicitly, whenever $2\nu/(\varepsilon t_0) > 1$,

$$N \leq \frac{2}{\delta}\sqrt{\nu}\log\left(\frac{2\nu}{\varepsilon t_0}\right) + 1$$

suffices.

Proof of Corollary 16.7. Since $0 < \delta/\sqrt{\nu} \leq 1$,

$$\log\left(1 + \frac{\delta}{\sqrt{\nu}}\right) \geq \frac{\delta}{2\sqrt{\nu}}.$$

Thus

$$t_N = t_0\left(1 + \frac{\delta}{\sqrt{\nu}}\right)^N \geq \frac{2\nu}{\varepsilon}$$

once

$$N\log\left(1 + \frac{\delta}{\sqrt{\nu}}\right) \geq \log\left(\frac{2\nu}{\varepsilon t_0}\right),$$

which is implied by the displayed bound. □

16.4 Feasible-Start Complexity

Theorem 16.8 (Conditional feasible-start bound for a general barrier)**.** *Let* Φ *be a* ν*-self-concordant barrier for* K*. Assume that* Φ *attains its minimum, and write*

$$\Phi^\star := \inf_{x \in \mathrm{int}(K)} \Phi(x).$$

Assume $p^\star > -\infty$ *and that the barrier subproblems* F_t *attain their minima. Assume that the objective range*

$$R_c := \sup_{x\in K}\langle c, x\rangle - \inf_{x\in K}\langle c, x\rangle$$

is finite. Let $x_{\mathrm{feas}} \in \mathrm{int}(K)$ *be given. If* $R_c = 0$*, then every feasible point is optimal. Otherwise, run the phase-switch Newton method defined in Theorem 15.10 on* Φ*, starting from* x_{feas}*, until the first iterate* x_{cen} *satisfying*

$$\lambda_\Phi(x_{\mathrm{cen}}) = \|\nabla\Phi(x_{\mathrm{cen}})\|_{x_{\mathrm{cen}},*} \leq \frac{1}{20}.$$

Then set

$$t_0 := \frac{1}{20R_c},$$

and run Algorithm 1 from (x_{cen}, t_0) *with* $\delta = 1/10$ *until* $t_N \geq 2\nu/\varepsilon$*. The output is strictly feasible and satisfies*

$$\langle c, x_N\rangle - p^\star \leq \varepsilon.$$

The total number of Newton steps is

$$O\Big(\Phi(x_{\text{feas}}) - \Phi^\star + 1 + \sqrt{\nu}\log\Big(1 + \frac{\nu R_c}{\varepsilon}\Big)\Big).$$

The constants $1/20$ *and* $1/10$ *are not special; they only leave enough room for the centered barrier gradient and the initial linear term* $t_0 c$ *to fit inside the* $1/10$*-Newton neighborhood.*

Proof of Theorem 16.8. By Lemma 16.1, the barrier Φ satisfies the domain-safety condition required by Theorem 15.10. Therefore the centering phase reaches $\lambda_\Phi(x_{\text{cen}}) \le 1/20$ after

$$O(\Phi(x_{\text{feas}}) - \Phi^\star + 1)$$

Newton steps; the final $+1$ absorbs the constant number of full Newton steps needed to pass from decrement $1/4$ to $1/20$.
At the centered point x_{cen}, Lemma 16.1 implies that

$$\|c\|_{x_{\text{cen}},*} \le R_c.$$

Indeed, for every h with $\|h\|_{x_{\text{cen}}} < 1$, both $x_{\text{cen}} + h$ and $x_{\text{cen}} - h$ belong to K, so

$$2\,|\langle c, h\rangle| \le R_c.$$

Applying this to ρh and letting $\rho \uparrow 1$ gives the displayed bound. Therefore

$$\lambda_{F_{t_0}}(x_{\text{cen}}) = \|\nabla\Phi(x_{\text{cen}}) + t_0 c\|_{x_{\text{cen}},*} \le \lambda_\Phi(x_{\text{cen}}) + t_0\,\|c\|_{x_{\text{cen}},*} \le \frac{1}{20} + \frac{1}{20} = \frac{1}{10}.$$

Thus (x_{cen}, t_0) is a valid warm start for Theorem 16.6 with $\delta = 1/10$. By Corollary 16.7, the path-following phase uses

$$O\Big(\sqrt{\nu}\log\Big(1 + \frac{\nu}{\varepsilon t_0}\Big)\Big) = O\Big(\sqrt{\nu}\log\Big(1 + \frac{\nu R_c}{\varepsilon}\Big)\Big)$$

Newton steps. Theorem 16.6 gives $\langle c, x_N\rangle - p^\star \le \varepsilon$, and strict feasibility follows from the Dikin ellipsoid inclusion and the self-concordant Newton steps used in the proof of Theorem 16.6. □

Corollary 16.9 (Dense arithmetic cost for LP under feasible-start assumptions)**.** *Consider the inequality-form linear program*

$$p^\star = \inf\left\{c^\top x : Ax \le b\right\}, \qquad A \in \mathbb{R}^{m\times n}, \quad c \in \mathbb{R}^n.$$

Assume that $K = \{x : Ax \le b\}$ *is bounded and full-dimensional, and that a strictly feasible point* x_{feas} *with* $Ax_{\text{feas}} < b$ *is given. Let*

$$\Phi(x) := -\sum_{i=1}^{m}\log(b_i - a_i^\top x), \qquad \Phi^\star := \inf_{Ax<b}\Phi(x),$$

where $a_i^\top$ *is the* i*th row of* A*. Define the objective range*

$$R_c := \max_{x\in K} c^\top x - \min_{x\in K} c^\top x.$$

If $R_c = 0$, then every feasible point is optimal. Otherwise, applying Theorem 16.8 to this logarithmic barrier returns a strictly feasible point with

$$c^\top x_N - p^\star \le \varepsilon$$

after

$$O\left(\Phi(x_{\text{feas}}) - \Phi^\star + 1\sqrt{m}\log\left(1 + \frac{mR_c}{\varepsilon}\right)\right)$$

Newton steps. Using dense linear algebra, each Newton step costs

$$O(mn^2 + n^3)$$

arithmetic operations. Hence the absolute-accuracy arithmetic complexity is

$$O\left(\left(\Phi(x_{\text{feas}}) - \Phi^\star + 1 + \sqrt{m}\log\left(1 + \frac{mR_c}{\varepsilon}\right)\right)(mn^2 + n^3)\right),$$

up to absolute numerical constants.

Proof of Corollary 16.9. For a bounded full-dimensional polytope, the logarithmic barrier Φ is an m-self-concordant barrier and attains its minimum at the analytic center. Therefore Theorem 16.8 applies with $\nu = m$. Since K is bounded, $R_c < \infty$, so the R_c-bound in Theorem 16.8 gives the stated Newton-step bound.
For the arithmetic count, a Newton step for the LP barrier uses

$$\nabla^2\Phi(x) = A^\top \operatorname{diag}((b - Ax)^{-2})A.$$

Forming this dense Hessian costs $O(mn^2)$, and solving the resulting $n \times n$ Newton system costs $O(n^3)$. Multiplying this per-step cost by the number of Newton steps proves the displayed complexity bound. ☐

Remark 16.3 (Comparison with the ellipsoid method for LP). For an LP $K = \{x : Ax \le b\}$, a separation oracle is cheap: check the m inequalities, and if $a_i^\top x > b_i$, return the violated halfspace $a_i^\top z \le b_i$. This costs $O(mn)$ arithmetic operations by dense matrix-vector multiplication. Combined with Corollary 6.11, the ellipsoid method uses roughly

$$O\left(n^2 \log\frac{R}{r}\right)$$

separation-oracle calls once suitable outer and inner radii are known. The short-step barrier method instead uses

$$O(\sqrt{m}\log(\cdots))$$

Newton steps, and each dense LP Newton step costs $O(mn^2 + n^3)$ arithmetic operations under Corollary 16.9. Thus ellipsoid gives the clean oracle route to polynomial-time LP, while the barrier method gives the Newton path-following mechanism. To turn the latter into a full polynomial-time LP algorithm, one still needs an initialization mechanism such as Phase I, an infeasible-start primal-dual method, or a homogeneous self-dual embedding.

Dependency and proof sketch

1. Definition 16.1 and Lemmas 16.1 and 16.2 turn the local self-concordant calculus of Lecture 15 into barrier geometry. The Dikin ellipsoid inclusion gives domain safety; the global barrier inequality gives accuracy.

2. Theorem 16.3 is the key optimization statement: centrality plus the global barrier inequality gives $\langle c, x(t)\rangle - p^\star \le \nu/t$.

3. Lemmas 16.4 and 16.5 are the two short-step mechanisms: increasing t moves the iterate to a slightly larger Newton neighborhood, and one Newton step maps it back to the δ-neighborhood.

4. Theorem 16.6 and Corollary 16.7 combine these two mechanisms into the $O(\sqrt{\nu}\log(\nu/\varepsilon))$ primal interior-point bound. Theorem 16.8 adds the feasible-start centering phase for a general barrier, and Corollary 16.9 specializes the conclusion to dense inequality-form LP arithmetic.

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