TTIC 31070: Convex Optimization
Homework 1

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Spring 2026

Problem 1 (Subgradients of concrete convex functions)**.** *Compute the subdifferentials in the following cases.*

(a) *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be*

$$f(x_1, x_2) := \max\{x_1, x_2\}.$$

Compute $\partial f(x_1, x_2)$ *for all* $(x_1, x_2) \in \mathbb{R}^2$*.*

(b) *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be*

$$f(x) := \|x\|_1 = \sum_{i=1}^{n} |x_i|.$$

Compute $\partial f(x)$ *for all* $x \in \mathbb{R}^n$*.*

(c) *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be*

$$f(x) := \|x\|_\infty = \max_{1 \le i \le n} |x_i|.$$

Compute $\partial f(x)$ *for all* $x \in \mathbb{R}^n$*.*

(d) *Let* $\|\cdot\|$ *be any norm on a finite-dimensional real vector space* E*, and let the dual norm on* E^* *be*

$$\|g\|_* := \sup_{\|x\| \le 1} \langle g, x \rangle .$$

Prove that

$$\partial \|\cdot\| (0) = \{g \in E^* : \|g\|_* \le 1\} ,$$

and that for every $x \neq 0$*,*

$$\partial \|\cdot\| (x) = \{g \in E^* : \|g\|_* = 1,\ \langle g, x \rangle = \|x\|\} .$$

Problem 2 (Jensen's inequality)**.** *Let* $C \subseteq E$ *be convex, and let* $f : C \to \mathbb{R}$ *be convex.*

(a) *Prove the finite Jensen inequality: if* $x_1, \ldots, x_m \in C$ *and* $\lambda_1, \ldots, \lambda_m \ge 0$ *with* $\sum_{i=1}^{m} \lambda_i = 1$*, then*

$$f\Big(\sum_{i=1}^{m} \lambda_i x_i\Big) \le \sum_{i=1}^{m} \lambda_i f(x_i).$$

(b) *Let* X *be a discrete random variable taking values in* C*, say* $X = x_i$ *with probability* λ_i*. Show that*

$$f(\mathbb{E}[X]) \le \mathbb{E}[f(X)].$$

Problem 3 (Carathéodory's theorem). *Let $S \subseteq \mathbb{R}^d$, and let $x \in \operatorname{conv}(S)$.*

(a) *Among all representations*

$$x = \sum_{i=1}^{m} \lambda_i s_i, \qquad s_i \in S,\ \lambda_i \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1,$$

choose one with m minimal. Prove that the points $s_1, \ldots, s_m$ must be affinely independent.

(b) *Deduce that $m \leq d + 1$.*

(c) *Conclude that every point in $\operatorname{conv}(S)$ can be written as a convex combination of at most $d+1$ points of S.*

Problem 4 (Log-sum-exp and softmax). *Define $F : \mathbb{R}^m \to \mathbb{R}$ by*

$$F(x) := \log\Bigl(\sum_{i=1}^{m} e^{x_i}\Bigr), \qquad x = (x_1, \ldots, x_m) \in \mathbb{R}^m.$$

(a) *For $i = 1, \ldots, m$, define*

$$\pi_i(x) := \frac{e^{x_i}}{\sum_{j=1}^{m} e^{x_j}}.$$

Show that $\pi_i(x) \geq 0$ and $\sum_{i=1}^{m} \pi_i(x) = 1$.

(b) *Compute $\nabla F(x)$ and show that*

$$\nabla F(x) = (\pi_1(x), \ldots, \pi_m(x)).$$

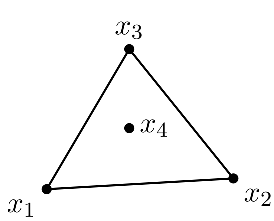
(c) *Deduce that F is convex on $\mathbb{R}^m$.*

(d) *Prove the bounds*

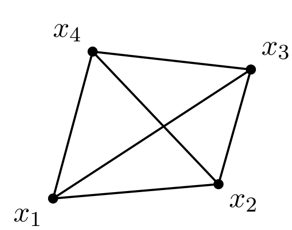
$$\max_{1 \leq i \leq m} x_i \leq F(x) \leq \max_{1 \leq i \leq m} x_i + \log m.$$

Problem 5 (Radon's theorem). *Let $x_1, \ldots, x_{d+2} \in \mathbb{R}^d$. Assume that the points are distinct for simplicity. The following form of Radon's theorem points in the opposite direction from Carathéodory's theorem: Carathéodory says that points in a convex hull admit sparse representations using at most $d+1$ points, whereas Radon says that once one has $d+2$ points in $\mathbb{R}^d$, the set can be partitioned into two disjoint subsets whose convex hulls intersect.*

In $\mathbb{R}^2$, Radon's theorem says that for any four points, either one point lies in the triangle formed by the other three, or the four points are in convex position and the two diagonals cross. Two typical pictures are:



(a) One point lies in the triangle formed by the other three.



(b) If the four points are in convex position, the two diagonals cross.

(a) *Prove that there exist scalars* $\alpha_1, \dots, \alpha_{d+2}$*, not all zero, such that*

$$\sum_{i=1}^{d+2} \alpha_i x_i = 0 \qquad \textit{and} \qquad \sum_{i=1}^{d+2} \alpha_i = 0.$$

(b) *Let*

$$I_+ := \{i : \alpha_i > 0\}, \qquad I_- := \{i : \alpha_i < 0\}.$$

Show that both sets are nonempty.

(c) *Define*

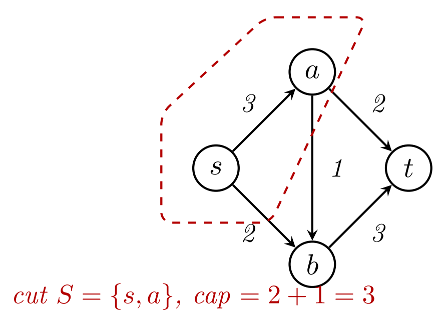
$$\lambda_i := \frac{\alpha_i}{\sum_{j \in I_+} \alpha_j} \quad (i \in I_+), \qquad \mu_i := \frac{-\alpha_i}{\sum_{j \in I_-} (-\alpha_j)} \quad (i \in I_-).$$

Show that $(\lambda_i)_{i \in I_+}$ *and* $(\mu_i)_{i \in I_-}$ *are convex coefficients and that*

$$\sum_{i \in I_+} \lambda_i x_i = \sum_{i \in I_-} \mu_i x_i.$$

(d) *Deduce that* $\{x_1, \dots, x_{d+2}\}$ *can be partitioned into two disjoint subsets whose convex hulls intersect.*

Problem 6 (Max-flow min-cut via LP duality)**.** *Let* $G = (V, E)$ *be a finite directed graph with source* s*, sink* t*, and edge capacities* $c_{uv} \geq 0$ *for each directed edge* $(u, v) \in E$*.*



For each vertex $v \in V$*, write*

$$E^+(v) := \{(v, w) \in E\}, \qquad E^-(v) := \{(u, v) \in E\}.$$

An s-t ***cut*** *is* $S \subseteq V$ *with* $s \in S$, $t \notin S$*; its capacity is*

$$\operatorname{cap}(S) := \sum_{\substack{(u,v) \in E \\ u \in S,\ v \notin S}} c_{uv}.$$

A feasible s-t ***flow*** *is a family*

$$f = (f_{uv})_{(u,v) \in E}$$

such that

$$0 \le f_{uv} \le c_{uv} \qquad ((u,v) \in E)$$

and

$$\sum_{e \in E^+(v)} f_e = \sum_{e \in E^-(v)} f_e \qquad (v \in V \setminus \{s,t\}).$$

Its value is

$$\mathrm{val}(f) := \sum_{e \in E^+(s)} f_e - \sum_{e \in E^-(s)} f_e.$$

(a) *Write the problem of maximizing* $\mathrm{val}(f)$ *over all feasible s-t flows as a linear program in the edge variables* $(f_{uv})_{(u,v)\in E}$.

(b) *Let* y_u *be the free dual variable for the flow conservation constraint at vertex* $v \in V \setminus \{s,t\}$ *and* z_{uv} *the nonnegative dual variable for the constraint* $f_{uv} \le c_{uv}$. *Show that the LP dual is*

$$\min \sum_{(u,v)\in E} c_{uv}\, z_{uv} \quad \text{s.t.} \quad y_s - y_t = 1, \qquad z_{uv} \ge \max(y_u - y_v,\, 0) \quad ((u,v) \in E).$$

(c) *Prove weak duality:*

$$\mathrm{val}(f) \le \sum_{(u,v)\in E} c_{uv}\, z_{uv}$$

for any feasible flow f *and any dual-feasible pair* (y,z).

(d) *Show that every s-t cut* S *gives a dual-feasible solution with objective value exactly* $\mathrm{cap}(S)$.

(e) *Let* (y,z) *be dual-feasible. Show that replacing* y *by* $y - y_t\mathbf{1}$ *preserves feasibility, so one may assume* $y_t = 0$ *and* $y_s = 1$. *Then, for* $\alpha \in [0,1)$, *let*

$$S_\alpha = \{v : y_v > \alpha\}.$$

Show that each S_α *is an s-t cut, and that*

$$\int_0^1 \mathrm{cap}(S_\alpha)\, d\alpha \;\le\; \textstyle\sum_{(u,v)\in E} c_{uv}\, z_{uv}.$$

Deduce that the LP dual optimum equals the minimum cut capacity, and that the dual admits an optimal $\{0,1\}$*-valued solution.*

(f) *Use strong LP duality to prove the max-flow min-cut theorem:*

$$\max\{\mathrm{val}(f) : f \text{ is a feasible s-t flow}\} = \min\{\mathrm{cap}(S) : S \text{ is an s-t cut}\}.$$